JOURNAL OF COMPUTATIONAL PHYSICS 60, 23–40 (1985)

A New Approach for Solving the Three-Dimensional Steady Euler Equations. I. General Theory*

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Received April 17, 1984; revised September 7, 1984

An iterative procedure is presented for solving steady inviscid three-dimensional (3-D) subsonic or incompressible rotational flow problems. The procedure combines concepts (Clebsch potentials and Munk–Prim substitution principle) from classical fluid mechanics with an extension to 3-D of a semi-direct Cauchy–Riemann solver.

INTRODUCTION

As is well known, the solution of the three-dimensional (3-D) steady Euler equations for subsonic or incompressible flows is a very difficult problem. This is because the flow equations are of hybrid type (locally neither elliptic nor hyperbolic) [1] and therefore cannot be directly solved using standard elliptic or hyperbolic solvers. In general, they are solved using time-marching methods. Unfortunately, due to the lack of dissipative mechanism, this classical approach may require very large computation time.

In the current study, a new method is developed for solving the above problem. It combines concepts from classical fluid mechanics with an extension to 3-D of a novel semi-direct Cauchy–Riemann solver [2]. In essence, the current method obtains the solution through the iterative execution of two interacting loops. The outer loop solves hyperbolic equations while the inner loop solves elliptic equations. The presentation of this work is divided into two parts. In Part I the theoretical foundation is developed while in Part II (the accompanying paper), several numerical results obtained using the current algorithm are presented to validate its usefulness in solving rotational flow problems.

For compressible flows, the governing differential equations which are assumed in this study describe a steady, inviscid, and non-heat-conducting flow in the absence of external body forces. The compressible fluid is a perfect gas with con-

* Partial results of this paper were presented at the AIAA 6th Computational Fluid Dynamics Conference, Danver, Mass., July 13–15, 1983.

0021-9991/85 $3.00

stant specific heat capacities. We will refer to the class of flows which are governed by the above equations as Munk–Prim flows to acknowledge the applicability of the Munk–Prim substitution principle [3] which significantly simplifies their solution. The details of this substitution principle and its application to the current problem are fully discussed in Section II. Briefly, it entails mapping an arbitrary Munk–Prim flow into a substitute flow which can be solved with less computational effort. Upon obtaining the substitute flow solution, the solution to the original problem is constructed with a reverse mapping procedure.

For incompressible flows, the governing differential equations which are assumed in this study describe a steady and inviscid flow. The fllow can be subjected to a conservative external body force. The solution procedure used is a simplified version of that for compressible flows.

The implementation of the Munk–Prim flow solver is presented on the flow chart shown in Fig. 1. The derivation of the equations solved in the two interacting loops

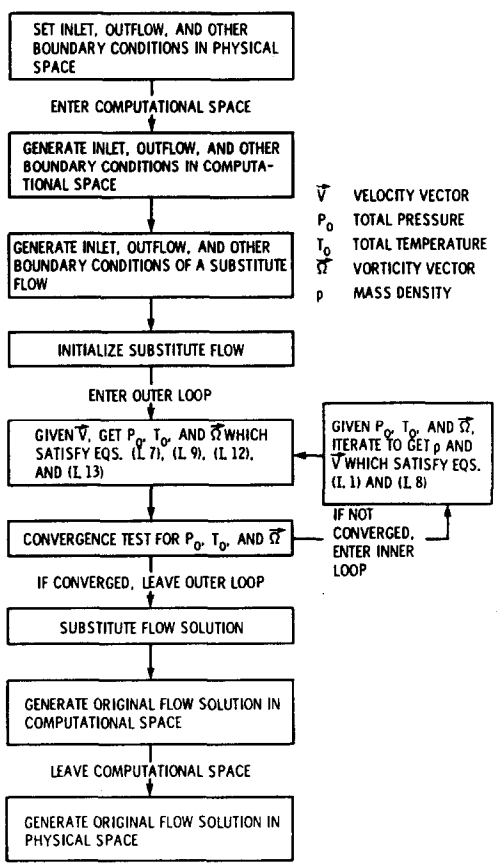

FIG. 1. Flow chart for Munk–Prim flow solver.

is presented in Section I. In the outer loop, the total pressure, total temperature, and vorticity distributions are found for a given velocity field. The solution strategy is presented in Section II and the questions of initial conditions and uniqueness of solution are addressed in Appendix A. An interesting feature of the current solution procedure for the vorticity vector is that it sidesteps solving the 3-D vorticity transport equation [4], in which the vorticity components are coupled. In the current procedure the vorticity components are obtained from the solutions of uncoupled partial differential equations (PDEs).

In the inner loop, the velocity and mass density fields are updated so as to satisfy the continuity equation and the condition that the curl of the velocity equals the vorticity. Within this loop, the total pressure, total temperature, and vorticity distributions, as well as a relationship between density and velocity, are assumed known. The iteration procedure by which the velocity and mass density fields are updated is presented in Section II and its stability is discussed in Appendix B.

I. Governing Equations

(I.1) *Munk–Prim Flows*

(I.1.1) *Vector Form*

The differential equations of the present analysis are the continuity equation

$$\vec{\nabla} \cdot (\rho \vec{V}) = 0 \tag{I.1}$$

momentum equation

$$(\vec{V} \cdot \vec{\nabla})\, \vec{V} + (1/\rho)\, \vec{\nabla} P = 0 \tag{I.2}$$

and entropy equation

$$\vec{V} \cdot \vec{\nabla} S = 0 \tag{I.3}$$

where $\vec{V}$, ρ, P, and S are the velocity vector, mass density, static pressure, and specific entropy, resectively. The above equations can be recast in terms of total pressure P_0, total temperature T_0, and velocity $\vec{V}$ by employing the following thermodynamic relationships:

$$\rho = \frac{P_0}{RT_0}\left(1 - \frac{\vec{V} \cdot \vec{V}}{2C_p T_0}\right)^{1/(\gamma - 1)} \tag{I.4}$$

$$P = P_0\left(1 - \frac{\vec{V} \cdot \vec{V}}{2C_p T_0}\right)^{\gamma/(\gamma - 1)} \tag{I.5}$$

and

$$S = C_p \ln T_0 - R \ln P_0. \tag{I.6}$$

Here C_p is the specific heat capacity at constant pressure, R the ideal gas constant, and γ the ratio of specific heat capacities. The parameters C_p, R, and γ are all constants for Munk–Prim flows. Throughout this study, the focus is to find the five variables P_0, T_0, and $\vec{V}$ (three components) which satisfy the five governing equations (Eqs. (I.1) to (I.3)). All other variables are considered functions of these fundamental variables.

It should be noted that Eqs. (I.1) to (I.6) can be nondimensionalized by scaling length, pressure, temperature, and density against a reference length L^*, a reference pressure P^*, a reference temperature T^*, and a reference density ρ^* (P^*, T^*, and ρ^* are assumed to satisfy the perfect gas law). Any additional reference variable is formed directly from P^*, T^*, and ρ^*. (For example, the dimensionless velocity $\stackrel{\text{def.}}{=} \vec{V}/\sqrt{P^*/\rho^*}$ and the dimensionless entropy $\stackrel{\text{def.}}{=} (S-S^*)/R$ with $S^* \stackrel{\text{def.}}{=} C_p \ln T^* - R \ln P^*$.) This results in a set of dimensionless flow equations which retain their original forms with a dimensionless gas constant $R=1$ and a dimensionless heat capacity $C_p = \gamma/(\gamma-1)$.

In the remainder of this subsection, we will collect the differential equations which are used in the two iteration loops which form the solver. To proceed, one notes that Eq. (I.2) and the thermodynamic relations can be combined to form Crocco's equation, i.e.,

$$\vec{V} \times \vec{\Omega} = \vec{\nabla} h_0 - T\vec{\nabla} S. \tag{I.7}$$

Here h_0 and T are the specific total enthalpy and absolute flow temperature, respectively. The vorticity vector $\vec{\Omega}$ is defined as

$$\vec{\Omega} = \vec{\nabla} \times \vec{V} \tag{I.8}$$

and thus

$$\vec{\nabla} \cdot \vec{\Omega} = 0. \tag{I.9}$$

In the current analysis Crocco's equation is used in place of the momentum equation. Also, since $\vec{V} \cdot \vec{V} \times \vec{\Omega} = 0$, Eqs. (I.3) and (I.7) imply that

$$\vec{V} \cdot \vec{\nabla} h_0 = 0. \tag{I.10}$$

With the aid of Eq. (I.6) and the relation

$$h_0 = C_p T_0 \tag{I.11}$$

Eqs. (I.3) and (I.10) can be replaced with the equivalent expressions:

$$\vec{V} \cdot \vec{\nabla} P_0 = 0 \tag{I.12}$$

and

$$\vec{V} \cdot \vec{\nabla} T_0 = 0. \tag{I.13}$$

In the inner iteration loop, P_0, T_0, and $\vec{\Omega}$ are provided by the outer loop and the objective is to find a velocity $\vec{V}$ (with corresponding density ρ from Eq. (I.4)), which satisfies Eqs. (I.1) and (I.8). The details of the inner loop iteration procedure are provided in Section II.

In the outer loop, the velocity field $\vec{V}$ is assumed known and the objective is to find a total pressure P_0, a total temperature T_0, and a vorticity $\vec{\Omega}$ which satisfy Eqs. (I.7), (I.9), (I.12), and (I.13). The evaluation of these variables can be divided into three steps. First, the unknowns P_0 and T_0 are solved using Eqs. (I.12) and (I.13). This can be done if the inlet distributions of P_0 and T_0 are known. In the second step, the variables h_0, T, and S are evaluated from the updated P_0 and T_0 values using Eqs. (I.6) and (I.11), and the relation: $T = T_0 - (\vec{V} \cdot \vec{V})/(2C_p)$. In the final step, the vorticity vector $\vec{\Omega}$, which is required to satisfy Eqs. (I.7) and (I.9), is constructed according to Theorem 2 of Appendix A, i.e.,

$$\vec{\Omega} = \vec{\nabla} h_0 \times \vec{\nabla} \tau + \vec{\nabla} S \times \vec{\nabla} \mu \tag{I.14}$$

where the two scalar functions τ and μ (referred to as the Clebsch potentials associated with $\vec{\Omega}$) satisfy the equations

$$\vec{V} \cdot \vec{\nabla} \tau = 1 \tag{I.15}$$

and

$$\vec{V} \cdot \vec{\nabla} \mu = -T. \tag{I.16}$$

It should be noted that $\vec{\Omega}$ is obtained directly from the solution of four uncoupled first order PDEs (Eqs. (I.12), (I.13), (I.15), and (I.16)). These hyperbolic PDEs have the same characteristics (i.e., streamlines) and thus can be integrated simultaneously. This procedure for calculating $\vec{\Omega}$ is far simpler than trying to solve Eqs. (I.7) and (I.9) directly. The solution of Eqs. (I.15) and (I.16) requires as input the inlet distributions of τ and μ. These distributions, according to Theorem 1 of Appendix A, must be chosen to be consistent with the streamwise vorticity at the inlet.

In the present analysis, the substitute flows are either homentropic ($\vec{\nabla} S = 0$) or homenergetic ($\vec{\nabla} h_0 = 0$). For homentropic flows, the second term on the right side of Eq. (I.14) vanishes. As a result, there is no need to solve Eq. (I.16). Furthermore, for this class of flows, T_0 becomes a function of P_0 (see Eq. (II.5)) which eliminates the need to solve Eq. (I.13). Thus only two PDEs, i.e., Eqs. (I.12) and (I.15), need to be solved in the outer loop for homentropic flows. Similarly, for homenergetic flows, only Eqs. (I.12) and (I.16) need to be solved. This observation justifies our earlier contention that the use of the Munk–Prim substitution principle can lead to a simpler procedure for solving the general Munk–Prim problem.

(I.1.2) *Tensor Form*

In the previous subsection, the basic equations for the two iteration loops were derived using vector notation. Before these equations are solved in a non-Cartesian

computational space, they shall be converted to general tensor forms. If we adopt Einstein's summation convention (which we shall use throughout this and the following papers), then in terms of the general computational coordinates $\{x^i\}$, the tensor forms of the outer loop Eqs. (I.12) to (I.16) are, respectively,

$$V^i \frac{\partial P_0}{\partial x^i} = 0 \tag{I.17}$$

$$V^i \frac{\partial T_0}{\partial x^i} = 0 \tag{I.18}$$

$$\sqrt{g}\, \Omega^i = \varepsilon^{ijk} \left(\frac{\partial h_0}{\partial x^j} \frac{\partial \tau}{\partial x^k} + \frac{\partial S}{\partial x^j} \frac{\partial \mu}{\partial x^k} \right) \tag{I.19}$$

$$V^i \frac{\partial \tau}{\partial x^i} = 1 \tag{I.20}$$

and

$$V^i \frac{\partial \mu}{\partial x^i} = -T. \tag{I.21}$$

In the above equations V^i is the contravariant velocity vector, Ω^i is the contravariant vorticity vector, g is the determinant of the covariant metric tensor g_{ij}, and ε^{ijk} is the contravariant Levi–Civita tensor density. It should be understood that the range of tensor indices, unless otherwise noted, is 1, 2, 3.

The tensor forms of the inner loop Eqs. (I.1) and (I.8) are given in Eqs. (II.7) to (II.9).

(I.2) *Incompressible Flows*

The analysis of the previous subsection encompasses incompressible flows as well. For these flows, the governing equations are Eqs. (I.1), (I.2), and

$$\rho = \text{constant}. \tag{I.22}$$

It should be noted that the flows governed by these equations can be subjected to a conservative external body force since the term associated with it can be absorbed into the pressure term [5]. These equations can also be nondimensionalized by introducing a reference length L^*, a reference density ρ^* ($\stackrel{\text{def.}}{=}$ fluid density), a reference velocity V^*, and a reference pressure P^* ($\stackrel{\text{def.}}{=} \rho^*(V^*)^2$). This results in a set of dimensionless flow equations which retain their original forms with a dimensionless density of $\rho = 1$.

Combining Eqs. (I.2), (I.8), and (I.22) yields

$$\vec{V} \times \vec{\Omega} = \vec{\nabla}(P_0/\rho) \tag{I.23}$$

with

$$P_0 \overset{\text{def.}}{=} P + \tfrac{1}{2}\rho(\vec{V}\cdot\vec{V}) = \text{total pressure.} \tag{I.24}$$

As a result of Eqs. (I.22) and (I.23), Eq. (I.12) is also valid for this special class of flows.

Equations (I.1), (I.8), (I.9), (I.12), and (I.23) are the equations to be solved for incompressible flows. Their solutions can be obtained in a fashion similar to that for homentropic flows.

II. Solution Strategy

(II.1) *Munk–Prim Substitution Principle and Its Application*

In Section I, it was shown that a Munk–Prim flow can be specified in terms of total pressure P_0, total temperature T_0, and the velocity vector $\vec{V}$. The Munk–Prim substitution principle states that, if P_0, T_0, and $\vec{V}$ specify a given Munk–Prim flow, then one can generate another Munk–Prim flow (referred to as the substitute flow) by the transformation

$$P_0' = P_0,\ T_0' = T_0/\alpha^2,\ \vec{V}' = \vec{V}/\alpha \tag{II.1}$$

where P_0', T_0', and $\vec{V}'$ are the flow variables of the substitute flow and α (Munk–Prim gauge factor) satisfies the equation

$$\vec{V}\cdot\vec{\nabla}\alpha = \vec{V}'\cdot\vec{\nabla}\alpha = 0. \tag{II.2}$$

The proof of this principle follows directly from Eqs. (I.1) to (I.6). From Eq. (I.5) and Eq. (II.1), it is seen that the static pressure is invariant under a Munk–Prim transformation. Similarly, the Mach number and the streamline pattern are also invariant under this transformation.

The substitution principle associated with Munk–Prim flows can be used to reduce the computational effort required to solve the general Munk–Prim flow problem. This reduction in effort is made possible by a proper choice of α. To explore this further, let α be equal to

$$\alpha = \alpha_1 \overset{\text{def.}}{=} \sqrt{T_0}\, P_0^{(1-\gamma)/2\gamma} \tag{II.3}$$

or

$$\alpha = \alpha_2 \overset{\text{def.}}{=} \sqrt{T_0}. \tag{II.4}$$

Both α_1 and α_2 satisfy Eq. (II.2) since T_0 and P_0 are constants along a streamline. If $\alpha = \alpha_1$, then Eqs. (II.1) and (II.3) coupled with Eq. (I.6) imply that

$$S' = 0 \qquad \text{and} \qquad T_0' = P_0'^{(\gamma-1)/\gamma} = P_0^{(\gamma-1)/\gamma} \tag{II.5}$$

i.e., the substitute flow is homentropic. Similarly, if $\alpha = \alpha_2$,

$$T_0' = 1 \qquad \text{and} \qquad P_0' = P_0 \tag{II.6}$$

i.e., the substitute flow is homenergetic. Solutions with either $S' = 0$ or $T_0' = 1$ require less work to evaluate since their generation requires two fewer equations to be solved per outer iteration cycle than that for the general flow problem.

As for the question of which choice of the Munk–Prim gauge factor has the advantage over the other, one recalls that Eqs. (I.12) and (I.15) need to be solved for the homentropic flow while Eqs. (I.12) and (I.16) need to be solved for the homenergetic flow. Since integration of Eq. (I.16), compared with that of Eq. (I.15), requires the extra effort of evaluating the temperature T, the choice with $\alpha = \alpha_1$ has a slight edge over the choice with $\alpha = \alpha_2$ in the outer loop calculations. However, the advantage shifts to the choice with $\alpha = \alpha_2$ in the inner loop calculations since evaluation of the density is slightly easier if $T_0 = 1$ (see Eq. (II.13)). Overall speaking, it is not clear which choice has the advantage over the other.

To generate the solution to the original problem from the solution of the substitute flow problem, the Munk–Prim gauge factor $\alpha_1(\alpha_2)$ must be evaluated throughout the flow field. This is accomplished by integrating Eq. (II.2). With α_1, (α_2) known, the solution to the original problem is found by employing Eq. (II.1).

(II.2) *Outer Loop*

In Section I, it was shown that only Eqs. (I.17) and (I.20) or Eqs. (I.17) and (I.21) need to be solved in the outer loop if a solution to a substitute flow problem is desired. These PDEs are first order and linear if the velocity field is assumed known. They may be solved using a standard marching procedure for hyperbolic equations. Once these equations are solved, the vorticity field is updated according to Eq. (I.19).

(II.3) *Inner Iteration Loop*

The basic equations which are solved in the inner loop are Eqs. (I.1) and (I.8). The tensor forms of these equations are, respectively,

$$\frac{\partial F^i}{\partial x^i} = 0 \tag{II.7}$$

and

$$\sqrt{g}\,\Omega^i = \varepsilon^{ijk}\frac{\partial V_k}{\partial x^j} \tag{II.8}$$

where F^i is the contravariant mass flux vector defined as

$$F^i = \rho\sqrt{g}\,g^{ij}V_j. \tag{II.9}$$

The variables which appear in Eqs. (II.8) and (II.9) and were not defined previously are V_i, the covariant velocity vector, and g^{ij}, the contravariant metric tensor. For Munk–Prim flows, the density is related to the velocity by Eq. (I.4). This equation, expressed in general tensor form, is

$$\rho = \frac{P_0}{RT_0}\left(1 - \frac{g^{ij}V_iV_j}{2C_pT_0}\right)^{1/(\gamma-1)} \tag{II.10}$$

Given P_0, T_0, and Ω^i, Eqs. (II.7) to (II.10) will be solved by an iterative procedure which is suggested by the work of Martin [2]. The equations which form the iterative procedure, unlike Eqs. (II.7) to (II.10), are valid only in a specified computational space and hence shall not be considered as tensor equations. However, in order to economize the mathematical presentation, we shall continue to use tensor notations and Einstein convention with one important exception, i.e., *the usual tensor analysis rules concerning the position (upper or lower) of indices will be disregarded.*

The iterative procedure consists of two schemes which are applied alternatively in odd and even numbered iteration cycles. Let n denote the iteration cycle number, then these two schemes (designated as Scheme A and Scheme B) are defined as follows:

SCHEME A. For $n = 1, 3, 5,\ldots$, $V_k^{(n)}$ (= the value of V_k at nth iteration) is required to satisfy

$$\varepsilon^{ijk}\frac{\partial V_k^{(n)}}{\partial x^j} = \sqrt{g}\,\Omega^i \tag{II.11}$$

and

$$\frac{\partial V_i^{(n)}}{\partial x^i} = \frac{\partial V_i^{(n-1)}}{\partial x^i}. \tag{II.12}$$

For $n = 1$, $V_i^{(0)}$ is equal to the covariant velocity at the end of the preceding outer loop cycle. For $n = 3, 5, 7,\ldots$, $V_i^{(n-1)}$ is obtained from the preceding Scheme B cycle (see Eq. (II.17)). The significance of Eqs. (II.11) and (II.12) and their solution procedure will be described shortly. For now, it suffices to note that, given $V_i^{(n)}$, one can define $\rho^{(n)}$ and $F^{i(n)}$ as

$$\rho^{(n)} \overset{\text{def.}}{=} \frac{P_0}{RT_0}\left(1 - \frac{g^{ij}V_i^{(n)}V_j^{(n)}}{2C_pT_0}\right)^{1/(\gamma-1)} \qquad (n = 1, 3, 5,\ldots) \tag{II.13}$$

and

$$F^{i(n)} \overset{\text{def.}}{=} \sqrt{g}\,g^{ij}\rho^{(n)}V_j^{(n)} \qquad (n = 1, 3, 5,\ldots). \tag{II.14}$$

SCHEME B. For $n = 2, 4, 6, \ldots$, $F^{i(n)}$ is required to satisfy

$$\frac{\partial F^{i(n)}}{\partial x^i} = 0 \qquad (n = 2, 4, 6, \ldots) \tag{II.15}$$

and

$$\varepsilon^{ijk} \frac{\partial F^{k(n)}}{\partial x^j} = \varepsilon^{ijk} \frac{\partial F^{k(n-1)}}{\partial x^j} \qquad (n = 2, 4, 6, \ldots). \tag{II.16}$$

Here $F^{k(n-1)}$ is given by Eq. (II.14). In terms of $F^{i(n)}$, $V_i^{(n)}$ is defined by

$$V_i^{(n)} \overset{\text{def.}}{=} g_{ij} F^{j(n)} / (\sqrt{g}\, \rho^{(n-1)}) \qquad (n = 2, 4, 6, \ldots). \tag{II.17}$$

As a result of Eq. (II.11), Scheme A effectively solves Eq. (II.8) while leaving Eq. (II.7) unsolved. Conversely, as a result of Eq. (II.15), Scheme B solves Eq. (II.7) while leaving Eq. (II.8) unsolved. If this iterative procedure converges, Eqs. (II.12) and (II.16) (they are introduced only to insure the uniqueness of $V_k^{(n)}$ and $F^{i(n)}$) become identities and the resulting solution must satisfy Eqs. (II.7) to (II.10).

The solution procedure for Eqs. (II.11) and (II.12) is different for $n = 1$ than for $n = 3, 5, 7, \ldots$. For $n = 1$, the first step is to find a vector ψ_i such that

$$\frac{\partial^2 \psi_i}{\partial x^j \partial x^j} = \varepsilon^{ijk} \frac{\partial V_k^{(0)}}{\partial x^j} - \sqrt{g}\, \Omega^i. \tag{II.18}$$

Since $V_i^{(0)}$ and Ω^i are known, Eq. (II.18) represents three Poisson's equations for the components of ψ_i. If the boundary conditions (BCs) of ψ_i are chosen such that

$$\frac{\partial \psi_i}{\partial x^i} = 0 \tag{II.19}$$

at all flow boundaries, then $V_i^{(1)}$ defined by

$$V_i^{(1)} \overset{\text{def.}}{=} V_i^{(0)} + \varepsilon^{ijk} \frac{\partial \psi_k}{\partial x^j} \tag{II.20}$$

represents a solution of Eqs. (II.11) and (II.12). To prove this assertion, one notes that Eq. (II.18) combined with

$$\frac{\partial (\sqrt{g}\, \Omega^i)}{\partial x^i} = 0 \tag{II.21}$$

(which is the tensor form of $\vec{\nabla} \cdot \vec{\Omega} = 0$) implies that

$$\frac{\partial^2}{\partial x^j \partial x^j} \left(\frac{\partial \psi_i}{\partial x^i} \right) = 0. \tag{II.22}$$

In other words, $(\partial\psi_i/\partial x^i)$ satisfies the Laplace's equation. As a result, Eq. (II.19) will be valid in the entire solution domain since it is satisfied at all flow boundaries. Upon combining Eqs. (II.18), (II.19), and (II.20), one concludes immediately that $V_i^{(1)}$ is a solution of Eqs. (II.11) and (II.12) for $n=1$.

To solve Eqs. (II.11) and (II.12) for $n=3, 5, 7,...$, one first obtains the solution to the Poisson's equation

$$\frac{\partial^2\sigma^{(n)}}{\partial x^i\,\partial x^i}=\frac{\partial}{\partial x^i}\left(V_i^{(n-1)}-V_i^{(1)}\right) \qquad (n=3, 5, 7,...). \tag{II.23}$$

Using the fact that $V_i^{(1)}$ is a solution of Eq. (II.11), it may be shown that $V_i^{(n)}$ defined by

$$V_i^{(n)} \overset{\text{def.}}{=} V_i^{(1)}+\frac{\partial\sigma^{(n)}}{\partial x^i} \qquad (n=3, 5, 7,...) \tag{II.24}$$

represents a solution of Eqs. (II.11) and (II.12). Similarly, to solve Eqs. (II.15) and (II.16) for $n=2, 4, 6,...$, one first obtains the solution of the Poisson's equation

$$\frac{\partial^2\varphi^{(n)}}{\partial x^i\,\partial x^i}=-\frac{\partial F^{i(n-1)}}{\partial x^i} \qquad (n=2, 4, 6,...) \tag{II.25}$$

and then constructs $F_i^{(n)}$ using the definition

$$F^{i(n)} \overset{\text{def.}}{=} F^{i(n-1)}+\frac{\partial\varphi^{(n)}}{\partial x^i} \qquad (n=2, 4, 6,...). \tag{II.26}$$

Except for $n=1$, all the inner loop iterations involve solving only one Poisson's equation (either Eq. (II.23) or Eq. (II.25)). The boundary conditions required to solve these Poisson's equations must be chosen to be consistent with the flow boundary conditions. The stability of this iterative procedure is discussed in Appendix B where it is shown to be a function of the metric tensor g_{ij}.

CONCLUSIONS

The current algorithm represents a new procedure for solving 3-D steady Euler equations. It is shown that the combined use of Clebsch potentials and the Munk–Prim substitution principle substantially reduces the computational effort of the outer loop.

In the inner loop, an iterative scheme is developed in which only Poisson's equations need to be solved. It will be shown in Part II that this scheme can be executed efficiently using fast Poisson's solvers.

APPENDIX A:
Basic Theorems of the Outer Loop

In the current study, the outer loop solves for the vorticity field $\vec{\Omega}$ which satisfies the equations

$$\vec{V} \times \vec{\Omega} = \vec{\nabla} h_0 - T\vec{\nabla} S \tag{A.1}$$

and

$$\vec{\nabla} \cdot \vec{\Omega} = 0. \tag{A.2}$$

Here, $\vec{V}$, h_0, T, and S are assumed to be known flow variables with h_0 and S satisfying the equations

$$\vec{V} \cdot \vec{\nabla} h_0 = 0 \tag{A.3}$$

and

$$\vec{V} \cdot \vec{\nabla} S = 0. \tag{A.4}$$

Given the above conditions, we establish the following theorems:

Theorem 1. *Assuming that the flow region is regular in the sense that every point in this region is either on the inlet surface or is connected to a point on the inlet surface through a well defined streamline, then a solution* $\vec{\Omega}$ *of Eqs.* (A.1) *and* (A.2) *is uniquely defined if the streamwise vorticity (i.e.,* $\vec{\Omega} \cdot \vec{e}_v$ *where* $\vec{e}_v \overset{\text{def.}}{=} \vec{V}/|\vec{V}|$*) is specified at the inlet surface.*

Proof. Let $\vec{\Omega}'$ and $\vec{\Omega}''$ be two solutions of Eqs. (A.1) and (A.2) with $\vec{\Omega}' \cdot \vec{e}_v = \vec{\Omega}'' \cdot \vec{e}_v$ at the inlet surface. If we define $\delta\vec{\Omega} = \vec{\Omega}'' - \vec{\Omega}'$, then one has

$$\vec{V} \times \delta\vec{\Omega} = 0 \tag{A.5}$$

$$\vec{\nabla} \cdot \delta\vec{\Omega} = 0 \tag{A.6}$$

and, at the inlet surface,

$$\delta\vec{\Omega} \cdot \vec{e}_v = 0. \tag{A.7}$$

Equation (A.5) coupled with the fact that $\vec{V} \neq 0$ (a streamline is not well defined at the point where $\vec{V} = 0$) implies that

$$\delta\vec{\Omega} = \sigma\vec{V} \tag{A.8}$$

where σ is a scalar function. Equations (A.6) to (A.8) imply that

$$\vec{V} \cdot \vec{\nabla}\sigma + (\vec{\nabla} \cdot \vec{V})\sigma = 0 \qquad \text{universally} \tag{A.9}$$

and

$$\sigma = 0 \qquad \text{at the inlet surface.} \tag{A.10}$$

Since the characteristics of PDE (A.9) are the streamlines which, by assumption, are well defined in the entire flow region, Eqs. (A.9) and (A.10) imply that $\sigma = 0$ universally [6]. As a result, $\delta\vec{\Omega} = \sigma\vec{V} = 0$ universally. Thus $\vec{\Omega}' = \vec{\Omega}''$. Q.E.D.

THEOREM 2. *Assuming that the flow region is regular and either* $\vec{\nabla}h_0 \neq 0$ *or* $\vec{\nabla}S \neq 0$ *throughout this region, then for any solution* $\vec{\Omega}$ *of Eqs.* (A.1) *and* (A.2), *there exist infinite many pairs of Clebsch potentials* τ *and* μ *such that* $\vec{\Omega}$ *can be expressed in a form given in Eq.* (I.14) *and* τ *and* μ *satisfy Eqs.* (I.15) *and* (I.16), *respectively. Conversely, for any* τ *and* μ *which satisfy Eqs.* (I.15) *and* (I.16), *Eq.* (I.14) *defines a solution* $\vec{\Omega}$ *of Eqs.* (A.1) *and* (A.2).

Proof. With the help of Eqs. (A.3), (A.4), (I.15), and (I.16), the second part of this theorem can be proved by directly substituting Eq. (I.14) into Eqs. (A.1) and (A.2). To prove the first part, first we assume that $\vec{\nabla}h_0 \neq 0$. For any solution $\vec{\Omega}$ of Eqs. (A.1) and (A.2), one introduces a vector field $\vec{\Omega}'$ such that

$$\vec{\Omega}' \equiv \vec{\Omega} - \vec{\nabla}S \times \vec{\nabla}\mu \tag{A.11}$$

Here μ is any solution of Eq. (I.16). Eqs. (A.1), (A.2), (A.4), (I.16), and (A.11) imply that

$$\vec{V} \times \vec{\Omega}' = \vec{\nabla}h_0 \tag{A.12}$$

and

$$\vec{\nabla} \cdot \vec{\Omega}' = 0. \tag{A.13}$$

As a result of Eq. (A.12), one has

$$\vec{\Omega}' \cdot \vec{\nabla}h_0 = 0. \tag{A.14}$$

Eq. (A.14) states that h_0 is a constant along any vortex line defined by $\vec{\Omega}'$. Using Eqs. (A.13) and (A.14), it is shown in Ref. [7] that there is a scalar function τ such that

$$\vec{\Omega}' = \vec{\nabla}h_0 \times \vec{\nabla}\tau, \tag{A.15}$$

provided $\vec{\nabla}h_0 \neq 0$. Substituting Eq. (A.15) into Eq. (A.12) and using Eq. (A.3), one obtains Eq. (I.15) as the condition for τ. Equation (I.14) follows directly from Eqs. (A.11) and (A.15). Since there are infinitely many solutions of Eq. (I.16), there must be infinitely many pairs of Clebsch potentials associated with any $\vec{\Omega}$ which satisfies Eqs. (A.1) and (A.2).

If $\vec{\nabla}h_0 = 0$, but $\vec{\nabla}S \neq 0$, a similar proof can be constructed to show that the first part of Theorem 2 is still valid. Q.E.D.

According to Theorems 1 and 2, there is freedom in the choice of the Clebsch potentials even if the inlet streamwise vorticity (and thus the entire vorticity field) is specified. This issue is discussed in Theorem 3.

THEOREM 3. *If τ and μ are a pair of Clebsch potentials associated with an $\vec{\Omega}$ which satisfies Eqs.* (A.1) *and* (A.2), *then in a region in which $\vec{\nabla}h_0 \times \vec{\nabla}S \neq 0$, τ' and μ' form another pair of Clebsch potentials associated with the same $\vec{\Omega}$ if and only if there is a function $\varphi(h_0, S)$ such that*

$$\tau' = \tau + \frac{\partial \varphi(h_0, S)}{\partial h_0} \tag{A.16}$$

and

$$\mu' = \mu + \frac{\partial \varphi(h_0, S)}{\partial S}. \tag{A.17}$$

Proof. By directly substituting Eqs. (A.16) and (A.17) into Eqs. (I.14) to (I.16) and using Eqs. (A.3) and (A.4), it can be shown that (τ', μ') and (τ, μ) are indeed associated with the same $\vec{\Omega}$ if they are related by Eqs. (A.16) and (A.17). To show the "only if" part of this theorem, first we define

$$\delta\tau = \tau' - \tau \qquad \text{and} \qquad \delta\mu = \mu' - \mu \tag{A.18}$$

Since, by assumption, both (τ', μ') and (τ, μ) are associated with the same $\vec{\Omega}$, Eqs. (I.14) to (I.16) and (A.18) imply that

$$\vec{\nabla}h_0 \times \vec{\nabla}\delta\tau + \vec{\nabla}S \times \vec{\nabla}\delta\mu = 0 \tag{A.19}$$

$$\vec{V} \cdot \vec{\nabla}\delta\tau = 0 \tag{A.20}$$

and

$$\vec{V} \cdot \vec{\nabla}\delta\mu = 0. \tag{A.21}$$

Using Eq. (A.19), one has

$$\vec{\nabla}S \cdot \vec{\nabla}h_0 \times \vec{\nabla}\delta\tau = 0 \tag{A.22}$$

and

$$\vec{\nabla}h_0 \cdot \vec{\nabla}S \times \vec{\nabla}\delta\mu = 0. \tag{A.23}$$

Equations (A.22) and (A.23) used in conjunction with the assumption that $\vec{\nabla}h_0 \times \vec{\nabla}S \neq 0$ imply that [8]

$$\delta\tau = f_1(h_0, S) \qquad \text{and} \qquad \delta\mu = f_2(h_0, S). \tag{A.24}$$

Here f_1 and f_2 are two arbitrary functions of h_0 and S. Substituting Eq. (A.24) into Eq. (A.19) and again invoking the assumption $\vec{\nabla}h_0 \times \vec{\nabla}S \neq 0$ yields

$$\frac{\partial f_1}{\partial S} = \frac{\partial f_2}{\partial h_0}. \tag{A.25}$$

As a result of Eq. (A.25), there exists a function $\varphi(h_0, S)$ such that

$$\delta\tau = f_1(h_0, S) = \frac{\partial \varphi(h_0, S)}{\partial h_0} \tag{A.26}$$

$$\delta\mu = f_1(h_0, S) = \frac{\partial \varphi(h_0, S)}{\partial S}. \tag{A.27}$$

$\delta\tau$ and $\delta\mu$ as given by Eqs. (A.26) and (A.27) automatically satisfy Eqs. (A.20) and (A.21) if Eqs. (A.3) and (A.4) are taken into account. Equations (A.16) and (A.17) follow directly from Eqs. (A.18), (A.26), and (A.27). Q.E.D.

Since $\vec{\nabla}h_0 \times \vec{\nabla}S = 0$ for both homentropic and homenergetic flows, Theorem 3 cannot be applied to these flows. Therefore the following theorems are given:

THEOREM 4. *In a region in which $\vec{\nabla}S = 0$ and $\vec{\nabla}h_0 \neq 0$, the function μ introduced in Theorem 2 need not be specified (see Eq. (I.14)). If τ is a Clebsch potential associated with an $\vec{\Omega}$ which satisfies equations (A.1) and (A.2), then τ' is another Clebsch potential associated with the same $\vec{\Omega}$ if and only if there is a function $\gamma(h_0)$ such that*

$$\tau' = \tau + \gamma(h_0). \tag{A.28}$$

THEOREM 5. *In a region in which $\vec{\nabla}h_0 = 0$ and $\vec{\nabla}S \neq 0$, the function τ introduced in Theorem 2 need not be specified. If μ is a Clebsch potential associated with an $\vec{\Omega}$ which satisfies Eqs. (A.1) and (A.2), then μ' is another Clebsch potential associated with the same $\vec{\Omega}$ if and only if there is a function $\theta(S)$ sucn that*

$$\mu' = \mu + \theta(S) \tag{A.29}$$

Theorems 4 and 5 can be proved using the same techniques employed in the Proof of Theorem 3. The details will not be presented.

APPENDIX B:
INNER LOOP STABILITY ANALYSIS

In this appendix, the stability of the inner loop iteration is analyzed assuming that the iterative increments of the density ρ can be neglected. This assumption makes a linear analysis possible and it may be valid for flows with low compressibility.

As in Section II, we shall use tensor notations and Einstein convention in this study to economize the mathematical expressions even though they may not be tensor expressions. Using Eqs. (II.12), (II.14), (II.17), and (II.23) to (II.26), it can be shown that, for $n = 4, 6, 8, \ldots$,

$$\frac{\partial^2(\sigma^{(n+1)} - \sigma^{(n-1)})}{\partial x^j \partial x^j} = \frac{\partial}{\partial x^i}\left(\frac{g_{ij}}{\rho\sqrt{g}}\frac{\partial \varphi^{(n)}}{\partial x^j}\right) \qquad (n = 4, 6, 8, \ldots) \tag{B.1}$$

and

$$\frac{\partial^2 \varphi^{(n+2)}}{\partial x^j \partial x^j} = -\frac{\partial}{\partial x^i}\left(\rho\sqrt{g}\, g^{ij}\frac{\partial}{\partial x^j}(\sigma^{(n+1)} - \sigma^{(n-1)})\right) + \frac{\partial^2 \varphi^{(n)}}{\partial x^j \partial x^j}$$
$$(n = 4, 6, 8, \ldots). \tag{B.2}$$

To study the stability of the inner loop iterations, we shall assume that

$$\lim_{n \to +\infty} \varphi^{(n)} = \varphi^{(\infty)} \qquad \text{and} \qquad \lim_{n \to +\infty} \sigma^{(n)} = \sigma^{(\infty)}. \tag{B.3}$$

Furthermore, it is assumed that

$$\varphi^{(n)} - \varphi^{(\infty)} = \mathrm{Re}(\bar{\varphi}^{(n)} e^{I(P_j x^j)}) \qquad (n = 2, 4, 6, \ldots) \tag{B.4}$$

$$\sigma^{(n)} - \sigma^{(\infty)} = \mathrm{Re}(\bar{\sigma}^{(n)} e^{I(P_j x^j)}) \qquad (n = 3, 5, 7, \ldots) \tag{B.5}$$

where P_j = the wave vector, $\bar{\varphi}^{(n)}$ and $\bar{\sigma}^{(n)}$ the constant amplitudes, Re = the real part of the quantity in parentheses, and $I = \sqrt{-1}$.

We shall also assume that the magnitude of P_i is so large and the corresponding wavelength so short that the coefficients ρ, g, g_{ij}, and g^{ij} are virtually constant in a small local region with dimensions comparable to the wavelength (i.e., the variations of the $(\varphi^{(n)} - \varphi^{(\infty)})$ and $(\sigma^{(n)} - \sigma^{(\infty)})$ are substantial in this region). Then, for any such local region, a substitution of Eqs. (B.3) to (B.5) into Eqs. (B.1) and (B.2) yields

$$\bar{\sigma}^{(n+1)} - \bar{\sigma}^{(n-1)} = \frac{g_{ij}}{\rho\sqrt{g}} \tilde{P}_i \tilde{P}_j \bar{\varphi}^{(n)} \qquad (n = 4, 6, 8, \ldots) \tag{B.6}$$

and

$$\bar{\varphi}^{(n+2)} = -\rho\sqrt{g}\, g^{ij} \tilde{P}_i \tilde{P}_j (\bar{\sigma}^{(n+1)} - \bar{\sigma}^{(n-1)}) + \bar{\varphi}^{(n)} \qquad (n = 4, 6, 8, \ldots). \tag{B.7}$$

Here $\tilde{P}_i$ are the direction cosines of the vector P_i, i.e.,

$$\tilde{P}_i \overset{\text{def.}}{=} \frac{P_i}{\sqrt{P_j P_j}} \tag{B.8}$$

and thus

$$\tilde{P}_i\tilde{P}_i = (\tilde{P}_1)^2 + (\tilde{P}_2)^2 + (\tilde{P}_3)^2 = 1. \tag{B.9}$$

Upon substituting Eq. (B.6) into Eq. (B.7), one obtains

$$\bar{\varphi}^{(n+2)} = G\bar{\varphi}^{(n)} \qquad (n = 4, 6, 8, \ldots) \tag{B.10}$$

where

$$G \overset{\text{def.}}{=} 1 - (g^{ij}\tilde{P}_i\tilde{P}_j)(g_{kl}\tilde{P}_k\tilde{P}_l) \tag{B.11}$$

As a result of Eq. (B.10), stability of the inner loop requires that

$$1 \geqslant |G|. \tag{B.12}$$

To study Eq. (B.12), we note that the matrices formed by g^{ij} and g_{kl} are positive definite hermitian matrices and are the inverse of each other. These properties are invariant under any coordinate transformation and, therefore, they must be valid in any computational space since the matrices formed by g^{ij} and g_{kl} in the physical space are both identity matrices. Assuming that λ_1, λ_2, and λ_3 ($\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 > 0$) are the eigenvalues of g_{kl} and $\tilde{P}_i$ is any vector satisfying Eq. (B.9), it can be shown that [9]

$$\frac{(\lambda_1 + \lambda_3)^2}{4\lambda_1\lambda_3} \geqslant (g^{ij}\tilde{P}_i\tilde{P}_j)(g_{kl}\tilde{P}_k\tilde{P}_l) \geqslant 1 \tag{B.13}$$

with the understanding that both bounds are sharp. Using Eq. (B.11) and inequality (B.13), it can be shown that the least upper bound of $|G|$ is

$$G_0 \overset{\text{def.}}{=} \frac{(\lambda_1/\lambda_3 - 1)^2}{4(\lambda_1/\lambda_3)}. \tag{B.14}$$

As a result, the stability condition (B.12) is reduced to

$$1 \geqslant G_0 \tag{B.15}$$

or, equivalently,

$$3 + 2\sqrt{2} \geqslant (\lambda_1/\lambda_3). \tag{B.16}$$

In other words, the ratio (λ_1/λ_3) is the only factor entering the stability consideration.

In conclusion, one notes that several limiting assumptions were made to linearize and localize the above stability analysis. It should also be noted that this analysis is intended only for the continuous version of the inner loop iterations. In spite of all these limitations, it will be shown in Part II that the results of this analysis do provide much useful information regarding the stability and the convergence rate of a discretized version of the inner loop iterations.

REFERENCES

1. J.-A. ESSERS, Time-Dependent Methods for Mixed and Hybrid Steady Flows," von Karman Institute for Fluid Mechanics, Lecture Series 1978-4.
2. E. D. MARTIN, "A Split-Recoupled-Semidirect Computational Technique Applied to Transonic Flow over Lifting Airfoils," AIAA Paper 78-11, Jan. 1978.
3. M. MUNK, AND R. C. PRIM, *Proc. Natl. Acad. Sci. USA* **33**, No. 5 (1947), 137–141.
4. A. VAZSONYI, *Quart. Appl. Math.* **3** (1945), 29–37.
5. G. K. BATCHELOR, "An Introduction to Fluid Dynamics," p. 508, Cambridge Univ. Press, New York/London, 1967.
6. R. COURANT, AND D. HILBERT, "Methods of Mathematical Physics," Vol. 2, Chap. 2, Interscience, New York, 1962.
7. H. LAMB, "Hydrodynamics," 6th ed., p. 248, Univ. Press, Cambridge, 1932.
8. H. B. PHILLIPS, "Vector Analysis," p. 41, Wiley, New York, 1933.
9. W. GREUB, AND W. RHEINBOLDT, *Proc. Amer. Math. Soc.* **11** (1959), 407–415.